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Minimum Polynomial for Single-Mode Parabose Parasupercharge and Hamiltonian

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Abstract We calculate the minimum polynomial $\phi(x, y)$ of parasupercharge Q and Hamiltonian H for single-mode parabose parasupersymmetry (P-PSUSY). Suppose that $\phi(x, y)$ satisfies the homogeneity $\forall\lambda \in \mathbb{R}, \phi(\lambda x, \lambda^2 y) = \lambda^n \phi(x, y)$, then the parafermionic order p_f is restricted to either 1, 2, or 4. Under the P-PSUSY, the homogeneity of the $\phi(x, y)$ is equivalent to the parasuperconformality of Q and H. The physical meaning of the parasuperconformality is discussed, in connection with the spin of the elementary particle.

Keywords Parasupersymmetry · Gröbner basis · Parasuperconformality · Neveu-Schwarz

1 Introduction

Supersymmetric quantum mechanics (SUSY QM) has many interesting applications.[1] For example, the Morse inequality can be derived from some SUSY QM Hamiltonian [2]. There are many generalizations to the SUSY QM. One of the natural generalizations is given by such that the fermion in the ordinary SUSY is replaced by parafermions, called parasupersymmetric (PSUSY) QM, which was first introduced by Rubakov and Spiridonov [3]. For the simplest case, where the parafermionic order $p_f = 2$, Q and H satisfy the cubic relation of $Q^3 = 4QH$.

It should be noticed that the cubic relation of $Q^3 = 4QH$ is the same as the relation where the superpotential is given by the harmonic oscillator potential (HOP), as is analogous to the ordinary SUSY, where the quadratic relation of $Q^2 = H$ for a general superpotential is the same as the relation in which the superpotential is given by HOP. Once the polynomial relation between Q and H is determined, the superpotential is not necessarily given by HOP.

However, the restriction to the HOP as a superpotential may be too strong to make a physically realizable PSUSY model. The reason is as follows. The ordinary bose commutation

[1] For a review of SUSY QM, including PSUSY, see, for example, [1].

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relation $[b, b^\dagger] = 1$ (for simplicity, we take a single mode into account) is realized as one of the irreducible representations of the $\mathbb{Z}_2$-graded $\mathfrak{sp}(2,\mathbb{R}) = \{c_-, c_+, h\}_\mathbb{R}$ by $h = \frac{1}{2}\{c_+, c_-\}$ and $c_\mp = \pm[c_\mp, h]$. Called a paraboson is the particle whose annihilator (creator) is isomorphic to the irreducible representation of c_- (c_+) in the $\mathbb{Z}_2$-graded $\mathfrak{sp}_2(2,\mathbb{R})$. Quantum field theory is compatible with the existence of parabosons [4] (although the parabosons have not yet observed on a fundamental level). In this sense, it is reasonable to take account of the PSUSY where the parabosonic potential is chosen as a superpotential. Such PSUSY may be called parabose PSUSY (P-PSUSY).

To distinguish the parabosonic state, the parabosonic order p_b is introduced, as analogously to the parafermionic order p_f, half of which amounts to the highest weight of $\mathfrak{su}(2)$. To obtain the polynomial relation of Q and H as $\psi(Q, H) = 0$, it is straightforward to use the Fock space representation such that H is diagonalized. However, it is not so easy a matter to examine whether or not the $\psi(x, y)$ is the minimum polynomial $\phi(x, y)$ of Q and H. Consider, for example, the degree of $\phi(x, y)$ with respect to x, denoted by $\deg_x \phi(x, y)$. Contrary to the intuitive expectation that $\deg_x \phi(x, y)$ may be given by $p_f + 1$, as is found in the ordinary SUSY (where $p_f = 1$) and in the $p_f = 2$ PSUSY, it is not necessarily true that $\deg_x \phi(x, y) = p_f + 1$. Due to the complicatedness of p_f and p_b dependence of $\phi(x, y)$, a comprehensive study of P-PSUSY has hardly been made, except the simplest case of $p_f = 2$.

As an application of P-PSUSY, we try to restrict the value of p_f under some reasonable condition. Suppose that the minimum polynomial $\phi(x, y)$ satisfies the homogeneity such that $\forall\lambda \in \mathbb{C}, \exists n \in \mathbb{N}, \phi(\lambda x, \lambda^2 y) = \lambda^n \phi(x, y)$. Then we obtain $p_f = 1, 2, 4$. Considering the isomorphism $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, we can interpret $p_f/2$ as a spin s (orbital angular momentum may be neglected, because we deal with a spatially one-dimensional case). Thus, $p_f = 1, 2, 4$ corresponds to $s = 1/2, 1, 2$, respectively. This result may be comparable to the spin of the elementary particles.

The aim of this paper is to first calculate the minimum polynomial $\phi(x, y)$ of Q and H for single-mode P-PSUSY, to unravel the p_f and p_b dependence of $\deg_x \phi(x, y)$. Then we show that under the P-PSUSY, the homogeneity of $\phi(x, y)$ is equivalent to the parasuperconformality of Q and H in the sense that Q, H and their generalization satisfy the generalized Neveu-Schwarz algebra. Finally, we discuss the physical meaning of the parasuperconformality.

Before proceeding to the calculation of the minimum polynomial in P-PSUSY, it should be remarked that there are other specific forms of PSUSY [5–8]. They are constructed in such a way that all the nonvanishing matrix elements of the fermionic operator f are chosen as the same, as in the form $(f)_{nm} \propto \delta_{n,m-1}$. This property, however, holds for a parafermion with $p_f = 1, 2$ only (see (5)), so that the minimum polynomial of Q and H in their work coincides with the one in the present work only for the case of $p_f = 1, 2$.

Moreover, it should be remarked that the P-PSUSY should not be confused with parabose-parafermi SUSY (PP-SUSY) [9–11], which is another generalization of the ordinary SUSY. In PP-SUSY, the relation of Q and H is the same as the ordinary SUSY, that is, $Q^2 = H$. To guarantee this relation, it is required that the parabosonic order p_b should be the same as the parafermionic order p_f (so that the ground state energy of H turns out to be vanishing, as in the ordinary SUSY), and that the parabosonic and parafermionic operators should not, in general, (anti)commute with each other. For the summary, see Table 1.

Table 1 Basic properties of SUSY and its generalizations: paraSUSY (PSUSY), parabose PSUSY (P-PSUSY), and parabose-parafermi SUSY (PP-SUSY) for a parabosonic order p_b and parafermionic order p_f. For the minimum polynomial, x and y correspond to the (para)supercharge Q and the Hamiltonian H, respectively. In the third column, super and parabose represent superpotential and parabosonic potential, respectively

Symmetry	(p_b, p_f)	Potential	Minimum polynomial	Commutativity
SUSY	$(1,1)$	Super	$x^2 - y$	$[b, f] = 0$
PSUSY	$(1,n)$	Super	$x(x^2 - 4y)$ $(p_f = 2)$	$[b, f] = 0$
P-PSUSY	(m,n)	Parabose	$x(x^2 - 4y)$ $(p_f = 2)$	$[b, f] = 0$
PP-SUSY	(n,n)	Parabose	$x^2 - y$	$[b, f] \neq 0$

2 Parabose ParaSUSY

2.1 Preliminaries

Let b (f) and $b^\dagger$ $(f^\dagger)$ denote the parabose (parafermi) annihilation and creation operators, respectively. Let P be the set of polynomials of b, f and $b^\dagger$, $f^\dagger$. Define the maps $\mathcal{A}, \mathcal{C} : P \to P$ by $\mathcal{A}(x) = \frac{1}{2}\{x^\dagger, x\}$ and $\mathcal{C}(x) = \frac{1}{2}[x^\dagger, x]$. Then the set $\{f, f^\dagger, \mathcal{C}(f)\}$ forms the 2-dimensional unitary algebra $\mathfrak{su}(2)$ (which is isomorphic to the rotation algebra $\mathfrak{so}(3)$), and the set $\{b, b^\dagger, \mathcal{A}(b)\}$ satisfies the $\mathbb{Z}_2$-graded $\mathfrak{sp}(2, \mathbb{R})$ algebra:

$$f = [f, \mathcal{C}(f)], \qquad b = [b, \mathcal{A}(b)], \tag{1}$$

and their Hermite adjoint. As is analogous to the ordinary SUSY, the parabose parasupercharge Q and the Hamiltonian H are given by

$$Q = \mathcal{Q} + \mathcal{Q}^\dagger, \qquad H = H_b + H_f, \tag{2}$$

where $\mathcal{Q} = (\mathcal{Q}^\dagger)^\dagger = b^\dagger f$, $H_b = \mathcal{A}(b)$, and $H_f = \mathcal{C}(f)$. Under the condition that Q is an integral of motion, that is

$$[Q, H] = 0, \tag{3}$$

it is sufficient to assume that $[b, H_f] = [f, H_b] = 0$. To guarantee these relations, we further assume that b is commutative with f and $f^\dagger$:

$$[b, f] = [b, f^\dagger] = 0. \tag{4}$$

The parabosonic order p_b (≥ 0) and parafermionic order $p_f \in \mathbb{N}$ are introduced in a usual context as $\mathfrak{z}\mathfrak{z}^\dagger|0\rangle = p_\mathfrak{z}|0\rangle$ (for $\mathfrak{z} = b, f$) with $|0\rangle$ representing the (unique) vacuum state such that $\mathfrak{z}|0\rangle = 0$. Let $|k-1\rangle_b$ and $|k-1\rangle_f$ denote the k-th normalized eigenstate of H_b and H_f, respectively $(k = 1, 2, \ldots)$. For given p_b and p_f, the irreducible representation of b and f can be written as [12–15]

$$\mathfrak{z}|k\rangle_\mathfrak{z} = c_k^{(\mathfrak{z})}|k-1\rangle_\mathfrak{z}, \qquad \mathfrak{z}^\dagger|k-1\rangle_\mathfrak{z} = c_k^{(\mathfrak{z})}|k\rangle_\mathfrak{z} \quad (\text{for } \mathfrak{z} = b, f), \tag{5}$$

where $c_{2k}^{(b)} = \sqrt{2k}$, $c_{2k+1}^{(b)} = \sqrt{2k + p_b}$, and $c_k^{(f)} = \sqrt{k(p_f + 1 - k)}$.

Under the above formulation, we will obtain a polynomial relation between Q and H (other than the trivial relation of (3)). By the construction of the P-PSUSY, it will be found in the next section that each coefficient of the polynomial can be chosen as a real number $\mathbb{R}$ for $p_b \geq 0$, so that we will have to obtain the polynomial $\phi(x, y) \in \mathbb{R}[x, y]$ such that $\phi(Q, H) = 0$, where $\mathbb{R}[x, y]$ represents the two-variable polynomial ring over $\mathbb{R}$. Due to (3), it is apparent that $\phi(x, y)$ belongs to a commutative ring over $\mathbb{R}$. There are many polynomial relations that Q and H satisfy, so that we will concentrate on a minimum polynomial. To obtain the minimum polynomial over $\mathbb{R}$, the following lemma is useful.

Lemma 1 *For $n \in \mathbb{N}$ and $f(x, y), g(x, y) \in \mathbb{R}[x, y]$,*

$$f^n(Q, H)g(Q, H) = 0 \quad \Longleftrightarrow \quad f(Q, H)g(Q, H) = 0.$$

Proof It is sufficient to prove for $n = 2$, due to the repeated application of the Lemma. It is trivial to show $\Longleftarrow$). Next, we show $\Longrightarrow$). For $n = 2$, we obtain $0 = f^2(Q, H)g^2(Q, H) = (f(Q, H)g(Q, H))^2 = h^\dagger h$, where $h = f(Q, H)g(Q, H)$. Here use has been made of $Q^\dagger = Q$, $H^\dagger = H$, $[Q, H] = 0$, and $f(x, y), g(x, y) \in \mathbb{R}[x, y]$. Noticing that $h^\dagger h$ is semi-positive and is vanishing if and only if $h = 0$, we get $h = 0$. □

Although the minimum polynomial $\phi(x, y)$ is uniquely determined except an overall scalar multiplication, we will fix the scalar multiplication in the following way. From the construction of $\phi(x, y)$, it will be found afterwards that $\phi(x, y)$ can be chosen as monic with respect to x. Once the $\phi(x, y)$ is chosen as x-monic, it is determined uniquely.

For later convenience of dealing with a parabose operators $b, b^\dagger$, we introduce an Hermitian operator R such that it anticommutes with b. Such an operator is well known, to be given by $R = 1 + 2\mathcal{C}(b)$. What is more characteristic of R is its eigenvalue: all the eigenvalues of R are given by $\pm(1 - p_b)$. This is confirmed from the irreducible representation of b. Rewriting R as $(1 - p_b)R_0$, we get

$$R_0^\dagger = R_0, \qquad \{R_0, b\} = 0, \qquad R_0^2 = 1. \tag{6}$$

2.2 Symmetry

Before calculating the minimum polynomial $\phi(x, y) \in \mathbb{R}[x, y]$, we examine a symmetry specific to the P-PSUSY. Consider the transformation $T : P \to P$ by

$$T : \begin{pmatrix} \mathfrak{z} \\ \mathfrak{z}^\dagger \end{pmatrix} \mapsto \begin{pmatrix} \mathfrak{z}' \\ (\mathfrak{z}^\dagger)' \end{pmatrix} = T_\mathfrak{z} \begin{pmatrix} \mathfrak{z} \\ \mathfrak{z}^\dagger \end{pmatrix} \quad (\text{for } \mathfrak{z} = b, f), \tag{7}$$

where T_b and T_f, which are 2×2 complex matrices, are chosen as such that the relation of (1) is invariant. Under the condition that H' (Q') is a linear combination of H_b and H_f (Q and $Q^\dagger$), we have two and only two transformations, denoted by Types A and B in Table 2.

In type A, the relation of $\mathfrak{z}\mathfrak{z}^\dagger|0\rangle = p_\mathfrak{z}|0\rangle$ with $\mathfrak{z}|0\rangle = 0$ remains invariant, so that the state vector $|n\rangle$ in the Fock space is invariant (up to an arbitrary phase); we have only to deal with the linear operators $\in P$. This implies that, by the Wigner's theorem [16], the transformation T may be given by a unitary transformation. Actually, T is given by the unitary transformation: $T : \mathfrak{z} \mapsto \mathfrak{z}' = U_\mathfrak{z}\mathfrak{z}U_\mathfrak{z}^\dagger$ (with $(\mathfrak{z}^\dagger)' = \mathfrak{z}'^\dagger$), where $U_\mathfrak{z} = e^{-i\theta_\mathfrak{z} H}$ ($\theta_\mathfrak{z} \in \mathbb{R}$) for $\mathfrak{z} = b, f$.

In Type B, however, the relation of $\mathfrak{z}\mathfrak{z}^\dagger|0\rangle = p_\mathfrak{z}|0\rangle$ with $\mathfrak{z}|0\rangle = 0$ does not remain. Thus, the state vector $|n\rangle$ should be transformed so that the theory be consistent. Notice that the spectrum of H_f is invariant as $\text{Spec}(H_f') = \text{Spec}(-H_f) = \text{Spec}(H_f) = \{-\frac{p_f}{2}, -\frac{p_f}{2} + 1, \ldots, \frac{p_f}{2}\}$ (while the spectrum of H_b is not invariant). This implies that the transformation

Table 2 Two types of transformation matrix $T_{\mathfrak{z}}$ for $\mathfrak{z}=b,f$, where $\theta=\theta_f-\theta_b$

Type	T_b	T_f	H'	Q'
A	$\begin{pmatrix} e^{i\theta_b} & 0 \\ 0 & e^{-i\theta_b} \end{pmatrix}$	$\begin{pmatrix} e^{i\theta_f} & 0 \\ 0 & e^{-i\theta_f} \end{pmatrix}$	H	$e^{i\theta}Q+e^{-i\theta}Q^\dagger$
B	$\begin{pmatrix} 0 & e^{-i\theta_b} \\ -e^{i\theta_b} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & e^{-i\theta_f} \\ e^{i\theta_f} & 0 \end{pmatrix}$	$-H$	$e^{i\theta}Q-e^{-i\theta}Q^\dagger$

of $\mathfrak{f}$ $(=f,f^\dagger)$ may be given by an (anti)unitary transformation. Actually, the transformation T of $\mathfrak{f}$ is given by the antiunitary transformation: $T:\mathfrak{f}\mapsto\mathfrak{f}'=\tilde{U}_f\mathfrak{f}\tilde{U}_f^\dagger$, where $\tilde{U}_f=KU_f$ with K defined by $K(c_1f|n\rangle+c_2f^\dagger|m\rangle)=\bar{c}_1f^\dagger|n\rangle+\bar{c}_2f|m\rangle$ (for $c_1,c_2\in\mathbb{C}$), so that K is antiunitary. Under the transformation $T:\mathfrak{f}\mapsto\mathfrak{f}'$, the parafermionic state $|n\rangle$ should be transformed as $T:|n\rangle\mapsto|n\rangle'=\tilde{U}_f|n\rangle$, so as to make the theory consistent. For $\mathfrak{z}=b$, on the other hand, T cannot be given by an (anti)unitary transformation, as is expected from the non-invariance of the spectrum of H_b as $\mathrm{Spec}(H_b')=\mathrm{Spec}(-H_b)=-\mathrm{Spec}(H_b)\neq\mathrm{Spec}(H_b)$. Actually, if b' and $b^{\dagger\prime}$ were given by $VbV^\dagger=e^{-i\theta_b}b^\dagger$ and $Vb^\dagger V^\dagger=-e^{i\theta_b}b$ (with $VV^\dagger=V^\dagger V=1$), respectively, we would have the relation of $0=e^{i\theta_b}$ $(\neq 0)$, a contradiction. The usefulness of the Type B transformation will be found in deriving, for example, the relation of (19).

Notice that while Q and H are not necessarily invariant under (7), $\mathcal{A}(f)$ and $\mathcal{C}(b)$ are invariant:

$$\mathcal{A}(f)\mapsto\mathcal{A}(f),\qquad \mathcal{C}(b)\mapsto\mathcal{C}(b),$$

so that $R\mapsto R$. In this sense, $\mathcal{A}(f)$ and $\mathcal{C}(b)$ (or R) may be regarded as more fundamental, compared to Q and H.

Proposition 2 *Denote by $\psi(Q,H)=0$ the (polynomial) relation between Q and H for the P-PSUSY. For $\mathbb{C}^*=\{z\in\mathbb{C}\,|\,|z|=1\}$, we have the following equivalence*:

$$\psi(Q,H)=0\quad\Longleftrightarrow\quad\forall\lambda\in\mathbb{C}^*,\quad\psi(\lambda Q\pm\lambda^{-1}Q^\dagger,\pm H)=0.$$

Proof For given p_b and p_f, the (polynomial) relation between Q and H is determined by the commutation relation of (1) (under the condition of (4)) only. Under the transformation in Table 2, the relations of (1) and (4) are invariant, so that we have $\psi(Q,H)=0\Longleftrightarrow\psi(Q',H')=0$. □

Remark 3 The function $\psi(x,y)$ in Proposition 2 is not necessarily chosen as a minimum polynomial $\phi(x,y)$. Note further that Proposition 2 does not necessarily hold for a general PSUSY. This is because for the P-PSUSY, there is a restriction on the parabose operator b as $b=[b,H_b]$, while for a general PSUSY, there is no such restriction on the parasuperpotential.

3 Minimum Polynomial

In this section, we calculate the minimum polynomial $\phi(x,y)$ for a general $p_f\in\mathbb{N}$ and $p_b\geq 0$, to find that $\deg_x\phi(x,y)$ is given by Table 3.

Table 3 $\deg_x \phi(x, y)$ for $p_f \in \mathbb{N}$ and ν $(:= p_b - 1)$

$p_f \in \mathbb{N}$	ν (≥ -1)	$\deg_x \phi(x, y)$
Even	Any real number	$p_f + 1$
Odd	$0, 2, 4, \ldots, p_f - 1$	$p_f + 1 + \nu$
	Otherwise	$2(p_f + 1)$

As the number $p_f \in \mathbb{N}$ increases, it tends to be more and more complicated to obtain the minimum polynomial $\phi(x, y)$ using the polynomial relations between f and $f^\dagger$ (together with $b = [b, H_b]$ and $R^2 = \nu^2$). Instead, we directly use the Fock space representation such that the Hamiltonian H is diagonalized. Denote by h_n the n-th eigenvalue of H. Recalling that the ground-state energies of H_b and H_f are given by $\frac{p_b}{2}$ and $-\frac{p_f}{2}$, respectively, we have

$$h_n = \frac{p_b}{2} - \frac{p_f}{2} + (n - 1) \quad (n = 1, 2, \ldots). \tag{8}$$

The eigenstates of H whose eigenvalue equals h_n are given by $|k\rangle_f |n - 1 - k\rangle_b$ $(k = 0, 1, \ldots, \mathrm{mim}(p_f, n - 1))$. As long as $n \geq p_f + 1$, the above eigenstates of H are $(p_f + 1)$-fold degenerate. Thus to obtain a polynomial relation between Q and H (not necessarily a minimum polynomial) as $\psi(Q, H) = 0$, we should obtain a polynomial $\psi(x, y)$ such that

$$\psi(Q, h_n)|k; n\rangle = 0, \quad (\text{for } k = 0, 1, \ldots, p_f), \tag{9}$$

where $|k; n\rangle := |k\rangle_f |n - 1 - k\rangle_b$. Here use has been made of $H|k; n\rangle = h_n |k; n\rangle$. It should be noticed that it is not necessary to calculate $\psi(Q, h_n)|k; n\rangle = 0$ for all k; it is sufficient to calculate it for $k = 0$ only.

Lemma 4 *For* $\psi(x, y) \in \mathbb{R}[x, y]$,

$$\psi(Q, h_n)|0; n\rangle = 0 \quad \Longrightarrow \quad \psi(Q, h_n)|k; n\rangle = 0 \quad (\textit{for } k = 0, 1, \ldots, p_f).$$

Proof Noticing inductively that $|k; n\rangle$ (for $k = 0, 1, \ldots, p_f$) can be written as a linear combination of $|0; n\rangle$, $Q|0; n\rangle, \ldots, Q^{p_f}|0; n\rangle$ over the real number, and using $[Q, \psi(Q, h_n)] = 0$, we find that $\psi(Q, h_n)|0; n\rangle = 0 \Longrightarrow \psi(Q, h_n)|k; n\rangle = 0$. □

Now we obtain the polynomial relation between Q and H as a function of p_b and p_f. First of all, we calculate $Q^2|k; n\rangle$, from which we obtain the following recurrence formula:

$$R_{k+1}^{(n)}|k; n\rangle = \sqrt{d_{k-1}^{(n)}}|k - 2; n\rangle + \sqrt{d_{k+1}^{(n)}}|k + 2; n\rangle \quad (\text{for } k = 0, 1, \ldots, p_f), \tag{10}$$

with the boundary condition $|-2; n\rangle = |-1; n\rangle = |p_f + 1; n\rangle = 0$, where $R_{k+1}^{(n)} = Q^2 - (e_k^{(n)} + e_{k+1}^{(n)})$ and $d_k^{(n)} = e_k^{(n)} e_{k+1}^{(n)}$, with $e_k^{(n)} = (c_k^{(f)} c_{n-k}^{(b)})^2$. Considering the boundary condition of $|p_f + 1; n\rangle = 0$, we find that for p_f even (or odd), the repeated application of (10) with $k = 1, 3, \ldots, p_f - 1$ (or $k = 0, 2, \ldots, p_f - 1$) yields the annihilator $\psi(Q, h_n) = \psi(Q, h_n; p_b, p_f)$ of $|1; n\rangle$ (or $|0; n\rangle$), such that

$$\psi(Q, h_n; p_b, p_f)|1 + p_f - 2\tilde{p}_f; n\rangle = 0, \quad \tilde{p}_f = \begin{cases} \frac{p_f}{2} & (\text{for } p_f \text{ even}), \\ \frac{p_f + 1}{2} & (\text{for } p_f \text{ odd}), \end{cases} \tag{11}$$

where $\psi(Q, h_n; p_b, p_f)$ written as

$$\psi(Q, h_n; p_b, p_f) = \sum_{k=0}^{[\tilde{p}_f/2]} (-1)^k \sum_{i \in S_k} (d_{i_1}^{(n)} d_{i_2}^{(n)} \dots d_{i_k}^{(n)})(R_{i_{k+1}}^{(n)} R_{i_{k+2}}^{(n)} \dots R_{i_{\tilde{p}_f - k}}^{(n)}), \tag{12}$$

with n in $d_i^{(n)}$, $R_i^{(n)}$ being related to h_n, p_b, p_f through (8). Here $[x]$ represents the integral part of x, and the set $S_k \in \mathbb{N}^{\tilde{p}_f - k}$ is given using the map $F : \mathbb{N}^{\tilde{p}_f - k} \to \mathbb{N}^{\tilde{p}_f - k}$ by $F(x) = 2x$ for p_f even (or $F(x) = 2x - 1$, with $1 = (1, 1, \dots, 1) \in \mathbb{N}^{\tilde{p}_f - k}$ for p_f odd) by

$$S_k = \bigcup_m F(m \oplus g(m)),$$

where $m = (m_1, m_2, \dots, m_k) \in \mathbb{N}^k$ with $1 \le m_1 \le m_2 - 2 \le \dots \le m_k - (2k-2) \le \tilde{p}_f - 1 - (2k-2)$, and $g(m) \in \mathbb{N}^{\tilde{p}_f - 2k}$ is constructed by eliminating from $(1, 2, 3, \dots, \tilde{p}_f) \in \mathbb{N}^{\tilde{p}_f}$ the elements of $m_1, m_1 + 1, \dots, m_k, m_k + 1$. Explicitly, for p_f odd

$$\begin{aligned} S_0 &= \{(1, 3, 5, \dots, p_f)\}, \\ S_1 &= \{(1, \check{3}, 5, 7, \dots, p_f), (3, 1, \check{5}, 7, \dots, p_f), (5, 1, 3, \check{7}, 9, \dots, p_f), \dots, \\ &\quad (p_f - 2, 1, 3, 5, \dots, p_f - 4, \check{p}_f)\}, \end{aligned}$$

and so on, where $\check{k}$ represents the elimination of k. For later convenience, it may be useful to expand $\psi(x, y; p_b, p_f)$ with respect to x and y as in the form

$$\psi(x, y; p_b, p_f) = \sum_{k=0}^{\tilde{p}_f} a_k(y) x^{2(\tilde{p}_f - k)}, \quad a_k(y) = \frac{\tilde{p}_f!}{(\tilde{p}_f - k)!} \sum_{i=0}^{k} \alpha_i^{(k)} y^i, \tag{13}$$

where $\alpha_0^{(0)} = 1$ and the coefficients $\alpha_i^{(k)}$ $(k = 1, 2, \dots)$ are to be calculated afterwards. The factor $1/(\tilde{p}_f - k)!$ is simply introduced to guarantee the vanishing of $a_k(y)$ for $k > \tilde{p}_f$.

Different from the functional form $c_k^{(b)}$ in (5) for k odd or even, the functional form of $\psi(Q, h_n; p_p, p_f)$ may be different for n in h_n being odd or even. Denote by $\psi_\pm(Q, h_n)$ the $\psi(Q, h_n; p_b, p_f)$ for n in h_n being $\{{}^{\text{odd}}_{\text{even}}\}$. Then the relation of (11) should be read as

$$\Psi(Q, h_n)|0, n\rangle = 0, \qquad \Psi(x, y) := \begin{cases} x \cdot \psi_+(x, y)\psi_-(x, y) & \text{(for } p_f \text{ even)}, \\ \psi_+(x, y)\psi_-(x, y) & \text{(for } p_f \text{ odd)}, \end{cases}$$

where use has been made of the relation of $Q|0; n\rangle \propto |1; n\rangle$. By Lemma 4 and (9), it is found that the minimum polynomial $\phi(x, y)$ of Q and H is given by the divisor of $\Psi(x, y)$. Noticing that $\psi_\pm(x, y)$ is monic with respect to x, we find that the minimum polynomial $\phi(x, y)$ can be chosen as x-monic.

Once $\phi(x, y)$ can be chosen as x-monic, it will be found that the x-degree of $\phi(x, y)$ (denoted by $\deg_x \phi(x, y)$) is restricted to such that

$$p_f + 1 \le \deg_x \phi(x, y) \le p_f + 1 + 2\tilde{p}_f. \tag{14}$$

First, we show $\deg_x \phi(x, y) \le p_f + 1 + 2\tilde{p}_f$. Recalling that $\phi(x, y)$ is given by the divisor of $F(x, y)$, we obtain $\deg_x \phi(x, y) \le \deg_x \Psi(x, y)$. Noticing that $\deg_x \psi_\pm(x, y) = 2\tilde{p}_f$, we get $\deg_x \phi(x, y) \le p_f + 1 + 2\tilde{p}_f$. Next, we show $p_f + 1 \le \deg_x \phi(x, y)$. To show this, it is convenient to use Proposition 2. Let n denote $\deg_x \phi(x, y)$. Expanding $\phi(\lambda\mathcal{Q} + \lambda^{-1}\mathcal{Q}^\dagger, H)$ with respect to λ, and recalling that $\phi(x, y)$ is chosen as x-monic, we find that the coefficient of λ^n is given by $\mathcal{Q}^n$, which should be vanishing by Proposition 2. Recalling that $b^\dagger$ is commutative with f and that $b^\dagger$ cannot be nilpotent, we find that the condition of $\mathcal{Q}^n = 0$ implies that $f^n = 0$. On the other hand, for a given p_f, f satisfies $f^{p_f+1} = 0$ with $f^{p_f} \neq 0$. This, together with the condition of $f^n = 0$ leads to $n \geq p_f + 1$ (if $n \leq p_f$, this would violate the condition of $f^{p_f} \neq 0$).

Now we are in a position to unravel the p_f and p_b-dependence of $\deg_x \phi(x, y)$. To begin with, we concentrate on the case of $p_b = 1$.

Lemma 5 *For P-PSUSY with $p_b = 1$, the x-degree of the minimum polynomial $\phi(x, y)$ is given by*

$$\deg_x \phi(x, y) = p_f + 1 \quad (\textit{for } p_b = 1).$$

Proof For $p_b = 1$ (or equivalently, $\nu = 0$), the functional form of $c_k^{(b)}$ is the same, despite k being odd or even. Thus it follows that $\psi_+(x, y) = \psi_-(x, y)$, so that the relation of (11) can be rewritten as

$$\Psi_0(Q, h_n)|0, n\rangle = 0, \qquad \Psi_0(x, y) := \begin{cases} x \cdot \psi_+(x, y) & (\text{for } p_f \text{ even}), \\ \psi_+(x, y) & (\text{for } p_f \text{ odd}). \end{cases}$$

In this case, the minimum polynomial $\phi(x, y)$ is given by the divisor of $\Psi_0(x, y)$, so that we get $\deg_x \phi(x, y) \leq \deg_x \Psi_0(x, y)$. Recalling that $\deg_x \psi_+(x, y) = 2\tilde{p}_f$, we obtain $\deg_x \Psi_0(x, y) = p_f + 1$. This, together with $p_f + 1 \leq \deg_x \phi(x, y)$ by (14), leads to $\deg_x \phi(x, y) = p_f + 1$. □

Considering that $\deg_x \phi(x, y) = \deg_x \Psi_0(x, y)$, we find that

$$\phi(x, y) = \Psi_0(x, y) \quad (\text{for } p_b = 1).$$

The next thing is to deal with the case of $p_b \neq 1$. Denote by $\hat{\phi}(x, y)$ the same functional form of the minimum polynomial for $p_b = 1$, that is, $\hat{\phi}_{p_f, p_b}(x, y) := \phi_{p_f, p_b=1}(x, y)$ (the symbol "hat" is introduced so that $\hat{\phi}(x, y)$ does not necessarily represent the minimum polynomial unless $p_b = 1$). Recalling that eigenvalues of the Hermitian operator R are given by $\pm(p_b - 1)$, we can reduce $\hat{\phi}(Q, H)$ to the form

$$\hat{\phi}(Q, H) = (R \text{ odd}) + (R \text{ even}), \tag{15}$$

where (R odd) and (R even) represent the polynomials of R such that is odd and even with respect to R, respectively. Recalling that $\hat{\phi}(x, y) \in \mathbb{Q}[x, y]$ by Lemma 5 and using (3), we find it trivial that the right-hand side of (15) should satisfy

$$X = X^\dagger, \qquad [Q, X] = [H, X] = 0, \tag{16}$$

where $X := (R \text{ odd}) + (R \text{ even})$.

Noticing from (13) that $\hat{\phi}(x, y)$ ($= \Psi_0(x, y)$) is x-odd for p_f even, or x-even for p_f odd, and using the repeated application of the identity $Q^2 = \mathcal{Q}^2 + \mathcal{Q}^{\dagger 2} + (1 - R)H_f + 2\mathcal{A}(f)H_b$, we can write $\hat{\phi}(Q, H)$ as $Q \cdot F(\mathcal{Q}^2 + \mathcal{Q}^{\dagger 2}, R, \mathcal{A}(f), H_b, H_c)$ for p_f even, or

$\tilde{F}(\mathcal{Q}^2 + \mathcal{Q}^{\dagger 2}, R, \mathcal{A}(f), H_b, H_c)$ for p_f odd, where F and $\tilde{F}$ represent (noncommutative) polynomial over $\mathbb{R}$ such that $F = F^\dagger$ and $\tilde{F} = \tilde{F}^\dagger$. Noticing further that $\mathcal{A}(f)$ can be written as the polynomial of H_f^2 over $\mathbb{R}$ (see Appendix), and using the commutativity $[R, H_b] = [R, \mathcal{Q}^2] = [R, \mathcal{A}(f)] = [R, H_f] = 0$ and $R^2 = \nu^2$, we can rewrite (R odd) and (R even) as

$$(R\ \text{odd}) = \begin{cases} Q \cdot F_1(Q^2, H_b, H_f; \nu^2)R & (\text{for } p_f \text{ even}), \\ \tilde{F}_1(Q^2, H_b, H_f; \nu^2)R & (\text{for } p_f \text{ odd}), \end{cases}$$

$$(R\ \text{even}) = \begin{cases} Q \cdot F_2(Q^2, H_b, H_f; \nu^2) & (\text{for } p_f \text{ even}), \\ \tilde{F}_2(Q^2, H_b, H_f; \nu^2) & (\text{for } p_f \text{ odd}), \end{cases}$$

where $F_i, \tilde{F}_i$ $(i = 1, 2)$ represent certain (noncommutative) polynomials over $\mathbb{R}$ such that $F_i = F_i^\dagger$, $\tilde{F}_i = \tilde{F}_i^\dagger$ and $QF_1 = (QF_1)^\dagger$, $QF_2 = (QF_2)^\dagger$.

3.1 For p_f Even

First, we deal with the case of p_f even. In this case, we have the following proposition:

Proposition 6 *For p_f even, we have*

$$\begin{cases} (R\ \text{odd}) = 0, \\ (R\ \text{even}) = \phi'(Q, H) \quad \textit{such that } \phi'(x, y) \in \mathbb{R}[x, y]. \end{cases}$$

Due to $\deg_x \phi'(x, y) < \deg_x \hat{\phi}(x, y)$, *we obtain*

$$\deg_x \phi(x, y) = p_f + 1 \quad (\textit{for } p_f \textit{ even}).$$

Proof First, we show that (R odd) $= 0$. Noticing the relation of $(QF_1R)^\dagger = R^\dagger(QF_1)^\dagger = RQF_1 = -QF_1R$, where use has been made of $\{R, Q\} = [R, F_1] = 0$, and recalling that $(QF_2)^\dagger = QF_2$, we find from $QF_1R + QF_2 = (QF_1R + QF_2)^\dagger$ by $X = X^\dagger$, that (R odd) $=$ 0. Next, we show that $F_2(Q^2, H_b, H_f; \nu^2)$ can be written as a polynomial of Q and H. From the condition of $[Q, F_2] = 0$ by $[Q, (R\ \text{even})] = 0$ (while the condition of $[H, F_2] = 0$ is automatically satisfied due to $[H, Q] = [H, H_b] = [H, H_f] = 0$), it is implied from $[Q, H_f] = \mathcal{Q} - \mathcal{Q}^\dagger$ and $[Q, H_b] = \mathcal{Q}^\dagger - \mathcal{Q}$, that the dependence of F_2 on H_b, H_f should can given by the combination of $H_b + H_f$, that is, F_2 can be written as a polynomial of Q and H, so can be (R even) (denoted by $\phi'(Q, H)$). Finally, we show that $\deg_x \phi'(x, y) < p_f + 1$ $(= \deg_x \hat{\phi}(x, y))$. Originally, ($R$ even) is derived from expanding Q^{p_f+1} with respect to $\mathcal{Q}, \mathcal{Q}^\dagger, H_b, H_f$, under the boundary condition of $\mathcal{Q}^{p_f+1} = (\mathcal{Q}^\dagger)^{p_f+1} = 0$. Thus Q-degree of (R even) turns out to be less than that of Q^{p_f+1}, that is, $\deg_x \phi'(x, y) < p_f + 1$. Eventually, for p_f even, the $\phi(x, y)$ is given by $\hat{\phi}(x, y) - \phi'(x, y)$, with $\deg_x \phi(x, y) = \deg_x \hat{\phi}(x, y) = p_f + 1$. □

Recalling that if $\psi_+(x, y) = \psi_-(x, y)$, as is realized in the case of $p_b = 1$, then $\deg_x \phi(x, y) = p_f + 1$, we expect the converse. This will be found to be true for p_f even, that is, the minimum polynomial $\phi(x, y)$ is given by $\Psi_0(x, y)$, as in the case of $p_b = 1$:

$$\phi(x, y) = \Psi_0(x, y) \quad (\text{for } p_f \text{ even}). \tag{17}$$

Table 4 Coefficients $\alpha_i^{(k)}$ for p_f even

k	i	$\frac{1}{(\tilde{p}_f+1)(2\tilde{p}_f+1)}\alpha_i^{(k)}$
1	0	0
	1	$-\frac{2}{3}$
2	0	$\frac{1}{105}\gamma$
	1	0
	2	$\frac{2}{45}(2\tilde{p}_f-1)(5\tilde{p}_f+6)$
3	0	0
	1	$-\frac{2}{945}\gamma\gamma'-\frac{64}{2205}(\tilde{p}_f+2)(\tilde{p}_f+3)(2\tilde{p}_f-1)(2\tilde{p}_f+3)$
	2	0
	3	$-\frac{4}{2835}(2\tilde{p}_f-1)(2\tilde{p}_f-3)(35\tilde{p}_f^2+91\tilde{p}_f+60)$

Actually, for p_f even, the coefficient $a_k(y)$ in (13) can be written as

$$\begin{cases} a_1(y) = -\sum_{i=1}^{\tilde{p}_f}\mathcal{S}_i, \\ a_2(y) = \sum_{1\le i<j\le\tilde{p}_f}\mathcal{S}_i\mathcal{S}_j - \sum_{i=1}^{\tilde{p}_f-1}\mathcal{P}_i, \\ a_3(y) = -\sum_{1\le i<j<k\le\tilde{p}_f}\mathcal{S}_i\mathcal{S}_j\mathcal{S}_k - \sum_{i=1}^{\tilde{p}_f-1}\mathcal{P}_i(\mathcal{S}_i+\mathcal{S}_{i+1}-\sum_{i=1}^{\tilde{p}_f}\mathcal{S}_i), \end{cases} \tag{18}$$

and so on, where $\mathcal{S}_i = e_{2i-1}^{(n)} + e_{2i}^{(n)}$ and $\mathcal{P}_i = e_{2i}^{(n)}e_{2i+1}^{(n)}$ (at the end of the calculation, n in $e_i^{(n)}$ should be replaced by $y-\frac{\nu-1}{2}+\tilde{p}_f$ by (8)). The double and triple summations can be rewritten using single summations as $\sum_{1\le i<j\le\tilde{p}_f}\mathcal{S}_i\mathcal{S}_j = \frac{1}{2}(Z_1^2-Z_2)$ and $\sum_{1\le i<j<k\le\tilde{p}_f}\mathcal{S}_i\mathcal{S}_j\mathcal{S}_k = \frac{1}{6}(Z_1^3-3Z_1Z_2+2Z_3)$, where $Z_k := \sum_{i=1}^{\tilde{p}_f}\mathcal{S}_i^k$ $(k=1,2,\dots)$. It is confirmed that $a_k(y)$ is uniquely determined, despite n in $e_k^{(n)}$ being odd or even. Explicitly, the coefficients $\alpha_i^{(k)}$ in $a_k(y)$ (see (13)) are summarized in Table 4, where $\gamma = (\tilde{p}_f+2)(9+7\nu^2-12\tilde{p}_f(1+\tilde{p}_f))$ and $\gamma' = (2\tilde{p}_f-3)(3\tilde{p}_f+5)+\frac{32}{7}(\tilde{p}_f+3)$.

From Table 4, it is expected that $\alpha_i^{(k)}=0$ for all $(k,i)\in(2\mathbb{N},2\mathbb{N}-1)$ and $(2\mathbb{N}-1,2\mathbb{N})$, which indicates that $a_k(y)$ be y-odd for k odd, or y-even for k even. This will be found to be true for p_f even, that is

$$a_k(-y) = (-1)^k a_k(y) \quad (\text{for } k=0,1,2\dots,\tilde{p}_f). \tag{19}$$

The relation of (19) is easily derived from Proposition 2. Choosing $\lambda = i$ and taking the lower sign, we find that the minimum polynomial $\phi(x,y)$ satisfies

$$\phi(ix,-y) = c\cdot\phi(x,y), \tag{20}$$

where $c = i^{p_f+1} = i(-1)^{\tilde{p}_f}$. This is because if $\phi(x,y)$ is the minimum polynomial of Q and H, then $\phi(ix,-y)$ is the minimum polynomial, and vice versa. Recall that for p_f even, the minimum polynomial $\phi(x,y)$ is given by $x\psi(x,y;p_b,p_f)$ with $\psi(x,y;p_b,p_f) = \sum_{k=0}^{\tilde{p}_f}a_k(y)x^{2(\tilde{p}_f-k)}$. Substituting this relation into (20) and comparing the coefficient of $x^{2(\tilde{p}_f-k)}$, we obtain (19).

Table 5 Coefficients $\alpha_i^{(k)}$ for p_f odd

k	i	$\alpha_i^{(k)}$
1	0	$\pm\frac{1}{2}\nu$
	1	$-\frac{1}{3}(4\tilde{p}_f^2-1)$
2	0	$\frac{1}{120}\nu^2(16\tilde{p}_f^3+16\tilde{p}_f^2-24\tilde{p}_f-9)-\frac{1}{70}(\tilde{p}_f+1)(4\tilde{p}_f^2-1)(4\tilde{p}_f^2-9)$
	1	$\mp\frac{1}{6}\nu(4\tilde{p}_f^2+4\tilde{p}_f+3)$
	2	$\frac{1}{90}(2\tilde{p}_f-3)(2\tilde{p}_f-1)(2\tilde{p}_f+1)(10\tilde{p}_f+7)$

3.2 For p_f Odd

Finally, we consider the case of p_f odd. Different from the case of p_f even, (R odd) in (15) is not vanishing unless $p_b = 1$, so that the right-hand side of (15) (denoted by X) cannot be written as a polynomial of Q and H. To obtain the polynomial relation between Q and H, we should eliminate R by squaring it, so that the x-degree of the minimum polynomial should be larger than $p_f + 1$. This implies that the functional form of $\psi(x, y; p_b, p_f)$ is not uniquely determined; otherwise, the minimum polynomial would be given by $\psi(x, y; p_b, p_f)$, where $\deg_x \psi(x, y; p_b, p_f) = p_f + 1$ is satisfied. Actually, for p_f odd, the coefficients $a_k(y)$ can be written as in a similar way to (18), where $\mathcal{S}_i, \mathcal{P}_i$ are replaced by $\mathcal{S}_i \to \mathcal{S}'_i = e^{(n)}_{2i-2} + e^{(n)}_{2i-1}$, $\mathcal{P}_i \to \mathcal{P}'_i = e^{(n)}_{2i-1} e^{(n)}_{2i}$ (at the end of the calculation, n in $e_i^{(n)}$ should be replaced by $y - \frac{\nu}{2} + \tilde{p}_f$, due to the relation of (8)). As a result, $a_1(y)$ and $a_2(y)$ are calculated, as is summarized in Table 5, where the complex sign comes from n in $e_i^{(n)}$ being $\{{}^{\text{odd}}_{\text{even}}\}$, respectively.

Notice that for $(k, i) \in (2\mathbb{N}, 2\mathbb{N}-1), (2\mathbb{N}-1, 2\mathbb{N})$, $\alpha_i^{(k)}$ is ν-odd. The reason is as follows. Suppose that (R odd) term is vanishing, as in the case of p_f even, the (R even) term can be written as a polynomial of Q and H, so that $\psi_+(x, y) = \psi_-(x, y)$. In this case, the minimum polynomial $\phi(x, y)$ is given by $\Psi_0(x, y)$, so that the relation of (19) should hold, as is analogous to the case of p_f even. Taking the contraposition of the above statement, we find that there is a (R odd) term from the fact that $\phi(x, y)$ is not given by $\Psi_0(x, y)$ for p_f odd.

Denote the greatest common divisor of $\psi_+(x, y)$ and $\psi_-(x, y)$ by $\gcd(\psi_+, \psi_-)$, which can be chosen as x-monic, due to the x-monicness of $\psi_\pm(x, y)$. Then by Lemma 1, the minimum polynomial is given by

$$\phi(x, y) = \frac{\Psi(x, y)}{\gcd(\psi_+, \psi_-)} \quad (\text{for } p_f \text{ odd}). \tag{21}$$

To obtain the $\gcd(\psi_+, \psi_-)$, it is convenient to calculate the Gröbner basis of $\psi_+(x, y)$ and $\psi_-(x, y)$. One of the basic properties of the Gröbner basis is that the common root of the original set of polynomials is the same as the common root of the corresponding Gröbner basis. If $\psi_+(x, y)$ and $\psi_-(x, y)$ have a common factor, there should exist a common root. Denote $\psi_\pm(x, y)$ by

$$\psi_\pm(x, y) = \psi^{(+)}(x, y) \pm \psi^{(-)}(x, y),$$

where $\psi^{(\pm)}(x, y) = \frac{1}{2}(\psi_+(x, y) \pm \psi_-(x, y))$. For $p_f = 1, 3, 5, 7$, the explicit form of $\psi^{(\pm)}(x, y)$ is summarized in Tables 6 and 7, with the Gröbner basis of $\psi_+(x, y)$ and $\psi_-(x, y)$ given in Table 8.

Table 6 $\psi^{(+)}$ for p_f odd. Note that $\psi^{(+)}$ is even with respect to ν

p_f	$\psi^{(+)}(x,y)$
1	x^2-y
3	$(x^2-y)(x^2-9y)-9+\frac{9}{4}\nu^2$
5	$(x^2-y)(x^2-9y)(x^2-25y)-36(9x^2-25y)+\frac{9}{4}\nu^2(11x^2-75y)$
7	$(x^2-y)(x^2-9y)(x^2-25y)(x^2-49y)-45[22x^2(3x^2-62y)+245(10y^2-9)]$ $+\frac{5}{2}\nu^2[x^2(47x^2-1478y)+2205(3y^2-5)]+\frac{(3\cdot5\cdot7)^2}{2^4}\nu^4$

Table 7 $\psi^{(-)}$ for p_f odd. Note that $\psi^{(-)}$ is odd with respect to ν

p_f	$\psi^{(-)}(x,y)$
1	$\frac{1}{2}\nu$
3	$\nu(x^2-9y)$
5	$\frac{3}{2}\nu[(x^2-9y)(x^2-25y)-300]+\frac{(3\cdot5)^2}{2^3}\nu^3$
7	$2\nu[(x^2-9y)(x^2-25y)(x^2-49y)-225(13x^2-245y)]+\frac{25}{2}\nu^3(17x^2-441y)$

Table 8 Gröbner basis of $\psi_+(x,y)$ and $\psi_-(x,y)$ with respect to x and y. Notice that G.B. $(\psi_+,\psi_-)=$ G.B. $(\psi^{(+)},\psi^{(-)})$

p_f	G.B. $(\psi_+(x,y),\psi_-(x,y))$
1	$\{\nu,\psi^{(+)}\}$
3	$\{\nu(\nu^2-4),\psi^{(-)},\psi^{(+)}\}$
5	$\{\nu(\nu^2-4)(\nu^2-16),\nu(\nu^2-4)(x^2-25y),\psi^{(-)},\psi^{(+)}\}$
7	$\{\nu(\nu^2-4)(\nu^2-16)(\nu^2-36),\nu(\nu^2-4)(\nu^2-16)(x^2-49y),$ $\nu(\nu^2-4)[4(x^2-25y)(x^2-49y)+245(\nu^2-36)],\psi^{(-)},\psi^{(+)}\}$

From Table 8, it is found that the Gröbner basis G.B. (ψ_+,ψ_-) can be written as in the form

$$\text{G.B.}\,(\psi_+,\psi_-)=\{\psi_0,\psi_1,\psi_2,\dots,\psi_{\tilde{p}_f}\},\tag{22}$$

where

$$\begin{cases}\psi_0=\prod_{n=1}^{(p_f-1)/2}(\nu^2-(2n)^2)\cdot\nu,\\ \psi_1=\prod_{n=1}^{(p_f-3)/2}(\nu^2-(2n)^2)\cdot\nu(x^2-p_f^2y),\\ \qquad\vdots\\ \psi_{\tilde{p}_f-1}=\psi^{(-)},\\ \psi_{\tilde{p}_f}=\psi^{(+)},\end{cases}$$

with ψ_k satisfying

$$\begin{cases}\psi_k\propto\prod_{n=1}^{\tilde{p}_f-1-k}(\nu^2-(2n)^2)\cdot\nu & (\text{for } k=0,1,\dots,\tilde{p}_f-1),\\ \deg_x\psi_k=2k & (\text{for } k=0,1,\dots,\tilde{p}_f).\end{cases}\tag{23}$$

Suppose that $\psi_+(x,y)$ and $\psi_-(x,y)$ have a common divisor, there should exist a common root, which is the same as the common root of the corresponding Gröbner basis. Thus from $\psi_0=0$, $\psi_+(x,y)$ and $\psi_-(x,y)$ have a common divisor, if and only if $\nu=0,2,4,\dots,p_f-1$. For $\nu=2k$ ($k=1,2,\dots,\tilde{p}_f-1$), it is found that $\psi_0,\psi_1,\dots,\psi_{\tilde{p}_f-1-k}$ are all vanishing, due to the first relation of (23). This, together with the second relation of (23), implies that $\mathrm{G.B.}(\psi_+,\psi_-)=\{\psi_{\tilde{p}_f-k}\}$, where we have used $\mathrm{G.B.}(\psi_+,\psi_-)=\mathrm{G.B.}(\psi_{\tilde{p}_f-k},\dots,\psi_{\tilde{p}_f})$. For ν otherwise, it follows that $\psi_0\neq 0$, so that $\mathrm{G.B.}(\psi_+,\psi_-)=\{1\}$. To summarize, we find that $\gcd(\psi_+,\psi_-)$, which is given by $\mathrm{G.B.}(\psi_+,\psi_-)$, can be written as

$$\gcd(\psi_+,\psi_-)=\begin{cases}\psi_{\tilde{p}_f-k} & (\text{for } \nu=2k \text{ with } k=0,1,\dots,\tilde{p}_f-1),\\ 1 & (\text{for } \nu \text{ otherwise}).\end{cases} \tag{24}$$

This, together with (21), leads to

$$\deg_x\phi(x,y)=\begin{cases}p_f+1+\nu & (\text{for } \nu=0,2,4,\dots,p_f-1),\\ 2(p_f+1) & (\text{for } \nu \text{ otherwise}).\end{cases}$$

However, the relation of (22) (with ψ_k satisfying (23)) is not rigorously derived for all odd p_f. This is just expected from Table 8. So we have the following problem:

Problem 7 *For P-PSUSY with p_f odd, show that* $\mathrm{G.B.}(\psi_+,\psi_-)$ *can be written as in* (22), *with ψ_k satisfying* (23).

As is found from Tables 6 and 7, it seems to be too complicated to obtain $\psi_\pm(x,y)$ for a general $p_f\in 2\mathbb{N}-1$. Thus, we should solve Problem 7 without calculating $\psi_\pm(x,y)$ explicitly.

4 Homogeneity and Conformality

4.1 Homogeneity

As an application of P-PSUSY, we try to restrict the spin degree of freedom, s, which is related to p_f as $s=p_f/2$. In ordinary quantum field theory, however, there is no restriction on the spin of the elementary particle; any integer or half-integer spin-s state can be represented by the set of $2s$ fermionic creation and annihilation operators. Suppose that the spin degree of freedom can be restricted in the context of P-PSUSY, then some reasonable condition should be imposed on the minimum polynomial $\phi(x,y)$.

Recall that as in the ordinary SUSY, the Hamiltonian H and the parasupercharge Q are commutative, so that there is a simultaneous eigenstate of H and Q (whose eigenvalues we denote by h and q, respectively). In the ordinary SUSY, H is given by the square of the supercharge Q, so that $h=q^2$ is satisfied. In the P-PSUSY, we assume an analogous relation: all the eigenvalues h are given by being proportional to q^2. In this case, the minimum polynomial $\phi(x,y)$ of Q and H can be written as $\phi(x,y)=\prod_i(x^2-\kappa_i y)$, where $\kappa_i\in\mathbb{R}$ represents some proportional constant, so that the $\phi(x,y)$ satisfies the homogeneity such that $\forall\lambda\in\mathbb{R}, \phi(\lambda x,\lambda^2 y)=\lambda^n\phi(x,y)$, where $n=\deg_x\phi(x,y)$ (recall that $\phi(x,y)$ can be chosen as x-monic). Denote by $\Phi^{(h)}$ the set of all the homogeneous polynomials such that $\Phi^{(h)}=\{f(x,y)\in\mathbb{R}[x,y] \mid \exists n\in\mathbb{N}, \forall\lambda\in\mathbb{R}, f(\lambda x,\lambda^2 y)=\lambda^n f(x,y)\}$. In

Table 9 Values of p_f, p_b for each element of $\Phi \cap \Phi^{(h)}$, where r represents any non-negative real number. The value of p represents the order of the paragrassmannian θ (see next subsection)

Element in $\Phi \cap \Phi^{(h)}$	(p_f, p_b)	p
$x^2 - y$	$(1, 1)$	1
$x(x^2 - 4y)$	$(2, r)$	2
$x(x^2 - 4y)(x^2 - 16y)$	$(4, 4)$	4

a similar way, denote by Φ the set of all the minimum polynomials for the P-PSUSY: $\Phi = \{\phi_{p_f, p_b}(x, y) | p_f \in \mathbb{N}, p_b \geq 0\}$.

Before discussing the physical meaning of the homogeneity of the minimum polynomial $\phi(x, y)$ for the P-PSUSY (which will be done in the next subsection), we will proceed to show that the homogeneity of $\phi(x, y)$ restrict the its functional form to such that

$$\Phi \cap \Phi^{(h)} = \{x^2 - y, x(x^2 - 4y), x(x^2 - 4y)(x^2 - 16y)\}. \tag{25}$$

For each element of $\Phi \cap \Phi^{(h)}$, the values of p_f and p_b are summarized in Table 9.

To show (25), it should be noticed that the homogeneity of the minimum polynomial $\phi(x, y)$ indicates the homogeneity of $\psi(x, y; p_b, p_f)$ as $\psi(\lambda x, \lambda^2 y; p_b, p_f) = \lambda^{2\tilde{p}_f} \psi(x, y; p_b, p_f)$. Thus for all $\lambda \in \mathbb{R}$, we get, using (13), the following equivalences:

$$\begin{aligned} \phi(\lambda x, \lambda^2 y) \equiv \lambda^n \phi(x, y) \quad &\Longleftrightarrow \quad a_k(\lambda^2 y) \equiv \lambda^{2k} a_k(y) \quad (k = 0, 1, \ldots, \tilde{p}_f) \\ &\Longleftrightarrow \quad \alpha_i^{(k)} = 0 \quad (i = 0, 1, \ldots, k-1; k = 1, \ldots, \tilde{p}_f). \end{aligned} \tag{26}$$

4.1.1 For p_f Even

First, we solve the last statement of (26) for $p_f \in 2\mathbb{N}$ (so that $\tilde{p} \in \mathbb{N}$). For $p_f = 2$ (or $\tilde{p}_f = 1$), we have only one requirement: $\alpha_0^{(1)} = 0$, which is automatically satisfied (see Table 4). Thus, the homogeneity of $\phi(x, y)$ is realized for $p_f = 2$. In this case, $\phi(x, y) = x(x^2 - 4y)$, despite the value of ν. For $p_f = 4$ (or $\tilde{p}_f = 2$), we have three requirements: $\alpha_0^{(2)} = \alpha_0^{(1)} = \alpha_1^{(2)} = 0$. The condition of $\alpha_0^{(2)} = 0$ leads to $\gamma = 0$. The remaining conditions of $\alpha_0^{(1)} = \alpha_1^{(2)} = 0$ are automatically satisfied. Thus, for $p_f = 4$, the homogeneity of $\phi(x, y)$ is realized for $p_f = 4$, where $\nu = 3$ is required by $\gamma = 0$, so that $\phi(x, y) = x(x^2 - 4y)(x^2 - 16y)$. For $p_f = 6, 8, \ldots$ (or $\tilde{p}_f = 3, 4, \ldots$), at least, another condition of $\alpha_1^{(3)} = 0$ is required, other than $\gamma = 0$ by $\alpha_0^{(2)} = 0$. However, $\alpha_1^{(3)}$ cannot be vanishing under the condition of $\gamma = 0$, so that the homogeneity of $\phi(x, y)$ cannot be realized for $p_f = 6, 8, \ldots$

4.1.2 For p_f Odd

For $p_f \in 2\mathbb{N} - 1$, we will make an analogous procedure to the case of $p_f \in 2\mathbb{N}$. For $p_f = 1$ (or $\tilde{p}_f = 1$), we have only one requirement: $\alpha_0^{(1)} = 0$, which leads to $\nu = 0$. Thus for $p_f = 1$, the homogeneity of $\phi(x, y)$ is realized under the condition of $\nu = 0$, in which $\phi(x, y)$ is given by $x^2 - y$. For $p_f = 3, 5, \ldots$ (or $\tilde{p}_f = 2, 3, \ldots$), another condition of $\alpha_0^{(2)} = 0$ is required, other than the condition of $\nu = 0$ by $\alpha_0^{(1)} = 0$. However, $\alpha_0^{(2)}$ cannot be vanishing under the condition of $\nu = 0$ (see Table 5), so that the homogeneity of $\phi(x, y)$ cannot be realized for $p_f = 3, 5, \ldots$

4.2 Conformality

In this subsection, the physical meaning of the homogeneity of $\phi(x,y)$ is discussed. Historically, the homogeneous polynomial relation between the general parasupercharge Q and Hamiltonian H was first introduced as a parasuperalgebra [17], which is given by generalizing the grassmannian variable θ in the ordinary superalgebra to the paragrassmannian such that $\theta^{p+1}=0$ (for $p\in\mathbb{N}$). Under the maps $H\mapsto\partial/\partial t$ and $Q\mapsto\partial/\partial\theta+\theta\cdot\partial/\partial t$ (in the Euclidean space with $t\in\mathbb{R}$), H and Q correspond to the superconformal generators L_1 and $G_{1/2}$, respectively. More generally, let L_m and G_r be given by

$$L_m = t^{-m+1}\frac{\partial}{\partial t}+\frac{1}{4}(1-m)t^{-m}\left(\theta\frac{\partial}{\partial\theta}-\frac{\partial}{\partial\theta}\theta\right),$$
$$G_r = t^{-r+1/2}\left(\frac{\partial}{\partial\theta}+\theta\frac{\partial}{\partial t}\right)+\frac{1}{2}(r-1/2)t^{-r-1/2}\left(\theta\frac{\partial}{\partial\theta}\theta-\theta^2\frac{\partial}{\partial\theta}\right),$$

then it is found that the generators L_m $(m\in\mathbb{Z})$ and G_r $(r=\mathbb{Z}+\frac{1}{2})$ satisfy [17]

$$[L_m,L_n]=(m-n)L_{m+n}\quad(\text{for all } p\in\mathbb{N}),\tag{27}$$

$$[L_m,G_r]=\frac{m-2r}{2}G_{m+r}\quad(\text{for all } p\in\mathbb{N}),\tag{28}$$

$$\begin{cases} G_rG_s-L_{r+s}+(\text{perm.})=0 & (p=1),\\ G_r(G_sG_t-4L_{s+t})+(\text{perm.})=0 & (p=2),\\ G_rG_sG_tG_u-10G_rG_sL_{t+u}+9L_{r+s}L_{t+u}+(\text{perm.})=0 & (p=3),\\ G_r(G_sG_tG_uG_v-20G_sG_tL_{u+v}+64L_{s+t}L_{u+v})+(\text{perm.})=0 & (p=4),\end{cases}\tag{29}$$

and so on, where (perm.) represents all the permutation with respect to $(r,s,\ldots)$. Notice that for $p=1$, L_m and G_r satisfy the (centerless) Neveu-Schwarz algebra, in which case, we may say that Q and H satisfy the superconformality.

Now we obtain for a general $p\in\mathbb{N}$, the polynomial relation of Q and H, which correspond to $G_{1/2}$ and L_1, respectively. By (28) and (29), the polynomial relation of $G_{1/2}$ and L_1 (more generally, G_r and L_{2r} for all $r\in\mathbb{Z}+\frac{1}{2}$) is given by, other than the commutativity $[G_r,L_{2r}]=0$, $\phi_p^{(c)}(G_r,L_{2r})=0$ such that

$$\phi_p^{(c)}(x,y)=\begin{cases}\prod_{n=0}^{(p-1)/2}(x^2-(p-2n)^2y) & (p\in2\mathbb{N}-1),\\ \prod_{n=0}^{p/2-1}(x^2-(p-2n)^2y)\cdot x & (p\in2\mathbb{N}).\end{cases}\tag{30}$$

It is apparent that $\phi_p^{(c)}(x,y)$ satisfies the homogeneity $\phi_p^{(c)}(\lambda x,\lambda^2y)=\lambda^{p+1}\phi_p^{(c)}(x,y)$ for all $\lambda\in\mathbb{R}$. Thus, it is expected that under the P-PSUSY, the parasuperconformality of Q and H be equivalent to the homogeneity of the minimum polynomial of Q and H in the sense that $\Phi\cap\Phi^{(c)}=\Phi\cap\Phi^{(h)}$, where $\Phi^{(c)}$ represents all the parasuperconformal polynomials: $\Phi^{(c)}=\{\phi_p^{(c)}(x,y)|p\in\mathbb{N}\}$. This expectation turns out to be true.

Proposition 8 $\Phi\cap\Phi^{(c)}=\Phi\cap\Phi^{(h)}$.

Proof Since $\phi_p^{(c)}(x,y)$ satisfies the homogeneity, it follows that $\phi_p^{(c)}(x,y)\in\Phi^{(h)}$, so that $\Phi^{(c)}\subset\Phi^{(h)}$. Hence we get $\Phi\cap\Phi^{(c)}\subset\Phi\cap\Phi^{(h)}$. Noticing that all the elements in $\Phi\cap\Phi^{(h)}$

(which is given by (25)) belong to $\Phi^{(c)}$ (see Table 9), we obtain $\Phi \cap \Phi^{(h)} \subset \Phi^{(c)}$. Hence, we find that $\Phi \cap \Phi^{(h)} = \Phi \cap (\Phi \cap \Phi^{(h)}) \subset \Phi \cap \Phi^{(c)}$. This, together with $\Phi \cap \Phi^{(c)} \subset \Phi \cap \Phi^{(h)}$, leads to $\Phi \cap \Phi^{(c)} = \Phi \cap \Phi^{(h)}$. □

However, the physical meaning of the parasuperconformality of Q and H is still not clear. Here we discuss the role of parasuperconformality of Q and H, focusing on the relation of $\phi_p^{(c)}(Q, H) = 0$. Denote by $h_n^{(p)}$ and $|h_n^{(p)}\rangle$ the n-th eigenvalue of H and the corresponding eigenstate for the paragrassmannian θ of order p, respectively. Recall that $[Q, H] = 0$, so that there is a simultaneous eigenstate of Q and H. For the simultaneous eigenstate such that the eigenvalue of H is given by $h_n^{(p)}$, the eigenvalue of Q (denoted by $q^{(p)}$) satisfies the polynomial equation of $\phi_p^{(c)}(q^{(p)}, h_n^{(p)}) = 0$. Since $\deg_x \phi_p^{(c)}(x, y) = p + 1$, it is found that the $q^{(p)}$ has $(p + 1)$ roots (denoted by $q_0^{(p)}, q_1^{(p)}, \ldots, q_p^{(p)}$), so that the H-eigenstate $|h_n^{(p)}\rangle$ is $(p+1)$-fold degenerate, as is similar to the $(p_f + 1)$-fold degeneracy of H for the P-PSUSY of parafermionic order p_f. To distinguish the degeneracy of $|h_n^{(p)}\rangle$, we introduce parameters k (for $k = 0, 1, \ldots, p$) such that $Q|h_n^{(p)}, k\rangle = q_k^{(p)}|h_n^{(p)}, k\rangle$ with $H|h_n^{(p)}, k\rangle = h_n^{(p)}|h_n^{(p)}, k\rangle$. Then, $|h_n^{(p)}\rangle$ can be written as a superposition of the eigenstates $|h_n^{(p)}, k\rangle$ (for $k = 0, \ldots, p$), that is, $|h_n^{(p)}\rangle \in \Omega_n^{(p)} := \{\sum_{k=0}^{p} c_k|h_n^{(p)}, k\rangle | (c_0, \ldots, c_p) \in \mathbb{C}^{p+1}\}$. For a given $p \in \mathbb{N}$, the Fock space $\Omega^{(p)}$ is given by a linear combination of all the eigenstates $|h_n^{(p)}\rangle$ (for $n = 1, 2, \ldots$):

$$\Omega^{(p)} = \left\{ \sum_{n=1}^{\infty} c_n |h_n^{(p)}\rangle | c_1, c_2, \ldots \in \mathbb{C} \right\}.$$

Noticing the relation of $\phi_p^{(c)}(x, y) = \phi_{p-2}^{(c)}(x, y)(x^2 - p^2 y)$, we can choose $q_k^{(p)}$ (for $k = 0, 1, \ldots, p - 2; k \neq p - 1, p$) as the roots of $\phi_{p-2}^{(c)}(x, h_n^{(p)}) = 0$. Define the subspace $\tilde{\Omega}^{(p)}$ of $\Omega^{(p)}$ by

$$\tilde{\Omega}^{(p)} = \left\{ \sum_{n=1}^{\infty} c_n |\tilde{h}_n^{(p)}\rangle | c_1, c_2, \ldots \in \mathbb{C} \right\} \subset \Omega^{(p)},$$

where $|\tilde{h}_n^{(p)}\rangle \in \tilde{\Omega}_n^{(p)} := \{\sum_{k=0}^{p-2} c_k|h_n^{(p)}, k\rangle | (c_0, \ldots, c_{p-2}) \in \mathbb{C}^{p-1}\} \subset \Omega_n^{(p)}$.

Suppose that the n-th eigenvalue $h_n^{(p)}$ of H is independent of the order p, that is

$$h_n^{(p)} = h_n^{(p-2)}.$$

(For the P-PSUSY, the corresponding relation can be realized for $p_b - p_f = \text{const.}$, see (8).) In this case, $q_k^{(p)}$ (for $k = 0, \ldots, p - 2$) are also the roots of $\phi_{p-2}^{(c)}(x, h_n^{(p-2)}) = 0$, so that the linear map $\varphi : \tilde{\Omega}^{(p)} \to \Omega^{(p-2)}$ over $\mathbb{C}$ by $\varphi(|h_n^{(p)}, k\rangle) = |h_n^{(p-2)}, k\rangle$ (for $n \in \mathbb{N}; k = 0, 1, \ldots, p - 2$) is commutative with Q and H:

$$\begin{array}{ccc} \tilde{\Omega}^{(p)} & \xrightarrow{Q,\,H} & \tilde{\Omega}^{(p)} \\ \downarrow \varphi & & \downarrow \varphi \\ \Omega^{(p-2)} & \xrightarrow{Q,\,H} & \Omega^{(p-2)}, \end{array} \tag{31}$$

provided that we should arrange the roots of $\phi_{p-2}^{(c)}(x, h_n^{(p-2)}) = 0$ as $q_k^{(p)} = q_k^{(p-2)}$ (for $k = 0, \ldots, p - 2$). The commutative diagram of (31) indicates that the spectrum of Q and H is invariant under the linear map φ, which is apparently bijective. In this sense, the subset $\tilde{\Omega}^{(p)}$

of $\Omega^{(p)}$ can be identified with $\Omega^{(p-2)}$, namely, $\tilde{\Omega}^{(p)} \cong \Omega^{(p-2)}$. Thus, $\Omega^{(p)}$ can be decomposed into

$$\begin{aligned}\Omega^{(p)} &= \Delta\Omega^{(p)} \oplus \tilde{\Omega}^{(p)} \\ &\cong \Delta\Omega^{(p)} \oplus \Omega^{(p-2)},\end{aligned} \tag{32}$$

where $\Delta\Omega^{(p)} := \Omega^{(p)} \setminus \tilde{\Omega}^{(p)}$. Notice that the degeneracy of the energy eigenstate in $\Delta\Omega^{(p)}$ is equal 2 $[= (p+1) - (p-1)]$ for all $p \geq 1$.

Here, it should be noticed that the degeneracy of the energy eigenstate in $\Omega^{(p)}$ (which amounts to $p+1$) is closely related to the spin degree of freedom for a massive particle. As is shown in Table 9, the value of p can be identified with p_f, half of which represents the highest weight of the irreducible representation of $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, so that $p+1$ may represent the angular momentum degree of freedom. Considering further that the orbital angular momentum may be neglected due to the one spatial dimensionality, we can regard $p+1$ as the spin degree of freedom $2s+1$ for a massive particle with spin s. In this sense, we have

$$p = p_f = 2s.$$

Denote by $\mathcal{F}^{(s)}$ and $\mathbf{F}^{(s)}$ a massive field with spin s and the corresponding Fock space generated by $\mathcal{F}^{(s)}$, respectively. For a free particle, the energy eigenstate $|n\rangle$ in $\mathbf{F}^{(s)}$ is $(2s+1)$-fold degenerate, as is the energy eigenstate in $\Omega^{(p)}$. Then we have the isomorphism:

$$\Omega^{(p)} \cong \mathbf{F}^{(s)} \quad (\text{for } p = 2s). \tag{33}$$

Substituting (33) into (32), we find that $\mathbf{F}^{(s)}$ can be decomposed into

$$\mathbf{F}^{(s)} \cong \Delta\mathbf{F}^{(s)} \oplus \mathbf{F}^{(s-1)}, \qquad \text{with } \Delta\mathbf{F}^{(s)} \subset \mathbf{F}^{(s)}. \tag{34}$$

This is the prediction of the parasuperconformality of Q and H: the decomposition of $\mathbf{F}^{(s)}$.

Suppose that $\mathcal{F}^{(s)}$ can be represented under the Lorentz group as

$$\mathcal{F}^{(s)} : \begin{cases} (\frac{s}{2}, \frac{s}{2}) & (\text{for } s \in \mathbb{N}), \\ (\frac{1}{2}, s - \frac{1}{2}) \oplus (s - \frac{1}{2}, \frac{1}{2}) & (\text{for } s \in \mathbb{N} + \frac{1}{2}), \end{cases}$$

as is the Dirac field ($s = 1/2$), the electro-weak gauge field ($s = 1$), the Rarita-Schwinger field ($s = 3/2$), and the (traceless) metric tensor ($s = 2$). In this case, the spin degree of freedom of the corresponding massless field (denoted by $\mathcal{F}_0^{(s)}$) turns out to be given by 2 (for $s \geq 1$), or 1 (for $s = 0, 1/2$). Recall that the spin degree of freedom of $\Delta\mathcal{F}^{(s)}$ is given by 2 $[= (2s+1) - (2s-1)]$ for $s \geq 1/2$, and notice the trivial inclusion relation of $\mathbf{F}_0^{(s)} \subset \mathbf{F}^{(s)}$. Then it seems reasonable that $\mathcal{F}_0^{(s)}$ may be chosen as a candidate of $\Delta\mathcal{F}^{(s)}$ for $s \geq 1$. More generally for $s \geq 1/2$, we have

$$\begin{cases} \Delta\mathcal{F}^{(s)} = \mathcal{F}_0^{(s)} & (\text{for } s \neq \frac{1}{2}), \\ \Delta\mathcal{F}^{(s)} \neq \mathcal{F}_0^{(s)} & (\text{for } s = \frac{1}{2}). \end{cases} \tag{35}$$

For $s = 1/2$, the relation of $\Delta\mathcal{F}^{(s)} \neq \mathcal{F}_0^{(s)}$ comes from the difference of the spin degree of freedom between $\Delta\mathcal{F}^{(s)}$ and $\mathcal{F}_0^{(s)}$; the spin degree of freedom of $\Delta\mathcal{F}^{(s)}$ is given by 2, while that of $\mathcal{F}_0^{(s)}$ is given by 1.

In the rest of this subsection, we discuss the physical meaning of the relation of (34) with (35). First, we consider the case of $s \in \mathbb{N}$. In this case, it will be shown that the relation of (34) with (35) can be interpreted as a (gauge) boson mass generalization through the Higgs mechanism.

For $s = 1$, $\mathcal{F}^{(s)}$ is chosen as one of the massive gauge bosons ($W^{\pm}, Z^0$). At high temperature where the electro-weak $\mathrm{U}(1) \times \mathrm{SU}(2)$ gauge symmetry is not broken, the gauge boson remains massless. Considering that the spin degree of freedom of the corresponding massless field $\mathcal{F}_0^{(s)}$ is given by 2, we can choose $\mathcal{F}_0^{(s)}$ as $\Delta\mathcal{F}^{(s)}$. In this case, $\mathcal{F}^{(s-1)}$ corresponds to one of the components $(\theta_1, \theta_2, \theta_3)$ of the original SU(2)-doublet complex scalar Higgs field $\mathcal{H} = \binom{\theta_1 + i\theta_2}{h + i\theta_3}$, where h, representing the standard Higgs boson, is not coupled with the photon, so that the photon remains massless.

For $s = 2$, we can make an analogous discussion to the case of $s = 1$. The mass of a (gauge) field $\mathcal{F}^{(s)}$ is generated by the coupling of the corresponding massless field $\mathcal{F}_0^{(s)}$ with the "vector Higg" field $\mathcal{F}^{(s-1)}$. If the coupling of $\mathcal{F}_0^{(s)}$ with $\mathcal{F}^{(s-1)}$ is null, as is the case of the photon for $s = 1$, the (gauge) field $\mathcal{F}_0^{(s)}$ may remain massless. This situation may be applied to the graviton, although the null-coupling Higgs $\mathcal{F}^{(s-1)}$ has not been specified yet.

Finally, we deal with the case of $s \in \mathbb{N} + \frac{1}{2}$. In this case, the physical meaning of the relation of (34) is less clear, compared to the case for $s \in \mathbb{N}$. However, in a practical case where the P-PSUSY is imposed on the conformal PSUSY, the value of s is restricted to either $1/2$, 1, or 2 (see Table 9). So we restrict ourselves to the case of $s = 1/2$ only. Different from the case of $s \neq 1/2$, $\mathbf{F}^{(s)}$ cannot be decomposed into $\mathbf{F}_0^{(s)}$ and $\mathbf{F}^{(s-1)}$, due to the vanishing of $\mathbf{F}^{(s-1)}$. This strongly suggests that for $s = 1/2$, the corresponding massless field $\mathcal{F}_0^{(s)}$ in itself not exit. Actually, all the leptons and quarks are observed to be massive.

5 Summary

We have calculated the minimum polynomial $\phi(x, y)$ of Q and H for single-mode P-PSUSY. For p_f even, the $\phi(x, y)$ can be written as in (17). For p_f odd, the $\phi(x, y)$ is given by (21), with $\gcd(\psi_+, \psi_-)$ written as (24) through the relation of (22). However, the relation of (22) has not been derived rigorously for all $p_f \in 2\mathbb{N} - 1$, so we have Problem 7. As p_f increase, $\psi_\pm(x, y)$ tends to be too complicated to deal with. Thus it is favorable to solve Problem 7 without obtaining $\psi_\pm(x, y)$ explicitly, although this has not been performed yet.

As an application of P-PSUSY, we have tried to restrict the value of p_f under some reasonable condition. Suppose that $\phi(x, y)$ satisfies the homogeneity $\forall\lambda \in \mathbb{R}, \phi(\lambda x, \lambda^2 y) = \lambda^n \phi(x, y)$ (where $n = \deg_x \phi(x, y)$), then we obtain $p_f = 1, 2, 4$. Considering that $p_f/2$ represents the highest weight of the irreducible representation of $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, we can regard $p_f + 1$ as the spin degree of freedom $(2s + 1)$ for a massive particle (with spin s), where the orbital angular momentum may be neglected due to the one spatial dimensionality. Thus, it follows that $s = 1/2, 1, 2$. Under the P-PSUSY, it has been shown that the homogeneity of the $\phi(x, y)$ is equivalent to the parasuperconformality of Q and H, where Q, H, and their generalization satisfy the generalized Neveu-Schwarz algebra.

The discussion on the physical meaning of the parasuperconformality here is summarized as follows. Under the parasuperconformality, the Fock space $\mathbf{F}^{(s)}$ of a massive spin-s field $\mathcal{F}^{(s)}$ can be decomposed as $\mathbf{F}_0^{(s)} \oplus \mathbf{F}^{(s-1)}$ for $s \neq 1/2$, where $\mathbf{F}_0^{(s)}$ represents the Fock space of the corresponding massless spin-s field $\mathcal{F}_0^{(s)}$. For $s = 1/2$, $\mathbf{F}^{(s)}$ cannot be decomposed into $\mathbf{F}_0^{(s)}$ and $\mathbf{F}^{(s-1)}$, due to the vanishing of $\mathbf{F}^{(s-1)}$, suggesting that there may not exist a corresponding massless field. For $s \in \mathbb{N}$, the above decomposition can be interpreted as a generalization of the (gauge) boson mass generation through the Higgs mechanism, where $\mathcal{F}^{(s-1)}$ plays a role of Higgs boson. If the coupling of $\mathcal{F}_0^{(s)}$ with $\mathcal{F}^{(s-1)}$ is null, $\mathcal{F}_0^{(s)}$ may remain massless. The above spin property is comparable to the spin of the elementary particle. For $s = 1/2$, all the leptons and quarks are massive. For $s = 1$, the masses of the W and Z bosons are generated by the Higgs mechanism, while the photon and gluon remain massless, due to the null coupling with the Higgs boson. For $s = 2$, there is a graviton $\mathcal{F}_0^{(s)}$, which is believed to be massless, although the null-coupling Higgs $\mathcal{F}^{(s-1)}$ with $\mathcal{F}_0^{(s)}$ has not been specified yet. It would be intriguing to construct a relativistic P-PSUSY model for $p_f = 4$.

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Appendix: Derivation of $\mathcal{A}(f) = g(H_f^2)$ ($g(x) \in \mathbb{R}[x]$)

By (5), the matrix element of $\mathcal{A}(f)$ and H_f^2 are given by

$$\begin{cases} [\mathcal{A}(f)]_{nm} = \delta_{nm}[(m + \frac{1}{2})p_f - m^2] \\ [H_f^2]_{nm} = \frac{1}{4}\delta_{nm}(p_f - 2m)^2 \end{cases} \quad (\text{for } n, m = 0, 1, \dots, p_f). \tag{36}$$

Noticing that $(m + \frac{1}{2})p_f - m^2$ and $(p_f - 2m)$ are invariant under the substitution of $m \to (p_f - m)$, we find that $\mathcal{A}(f)$ and H_f^2 can be decomposed into

$$\begin{cases} \mathcal{A}(f) = A \oplus \bar{A} \\ H_f^2 = B \oplus \bar{B} \end{cases} \quad (A, B \in \mathcal{D}_{(p_f+1)/2} \text{ for } p_f \text{ odd}),$$

and

$$\begin{cases} \mathcal{A}(f) = A \oplus a_0 \oplus \bar{A} \\ H_f^2 = B \oplus 0 \oplus \bar{B} \end{cases} \quad (A, B \in \mathcal{D}_{p_f/2} \text{ for } p_f \text{ even}),$$

where $a_0 = \frac{p_f}{2} + \frac{p_f^2}{4}$, $\mathcal{D}_n$ represents the set of $n \times n$ diagonal matrices over $\mathbb{R}$, and the map $^{-}: \mathcal{D}_n \to \mathcal{D}_n$ represents the linear map such that for $\mathcal{D}_n \ni X = \operatorname{diag}(a_1, a_2, \dots, a_n)$, $\bar{X} = \operatorname{diag}(a_n, \dots, a_2, a_1)$.

1. For p_f odd, $B, B^2, \dots, B^{(p_f+1)/2} \in \mathcal{D}_{(p_f+1)/2}$ are linear independent over $\mathbb{R}$, and they form a complete set in $\mathcal{D}_{(p_f+1)/2}$. Thus, $A \in \mathcal{D}_{(p_f+1)/2}$ can be written as a linear combination of $B, B^2, \dots, B^{(p_f+1)/2}$, that is, $A = \sum_{n=1}^{(p_f+1)/2} c_k B^k$ ($c_k \in \mathbb{R}$). In this case, $\mathcal{A}(f)$ can be written as $\sum_{n=1}^{(p_f+1)/2} c_k (H_f^2)^k$, due to $(\bar{B})^k = \overline{B^k}$.
2. For p_f even, $(B \oplus 0), (B \oplus 0)^2, \dots, (B \oplus 0)^{p_f/2}$; and $\mathbb{1} \oplus 1 \in \mathcal{D}_{p_f/2} \oplus \mathcal{D}_1$ are linear independent over $\mathbb{R}$, and they are complete in $\mathcal{D}_{p_f/2+1}$. Thus for $A \oplus a_0 \in \mathcal{D}_{p_f/2+1}$, we can write $A \oplus a_0 = c_0(\mathbb{1} \oplus 1) + \sum_{n=1}^{p_f/2} c_k (B \oplus 0)^k$, from which we obtain $A = c_0\mathbb{1} + \sum_{n=1}^{p_f/2} c_k B^k$ and $a_0 = c_0$. In this case, $\mathcal{A}(f)$ can be written as $a_0\mathbb{1}' + \sum_{n=1}^{p_f/2} c_k (H_f^2)^k$, where $\mathbb{1}' = \mathbb{1} \oplus 1 \oplus \mathbb{1}$.

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